

GALOIS LINES FOR SPACE ELLIPTIC CURVE WITH $j = 12^3$

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ABSTRACT. The V_4 -lines for each linearly normal space elliptic curve form the edges of a tetrahedron, however in case the elliptic curve has $j = 12^3$, there exist Z_4 -lines. We show the arrangement of V_4 and Z_4 -lines explicitly for such a curve. As a corollary we obtain that each irreducible quartic curve with genus one has at most two Galois points.

MSC: primary 14H50, secondary 14H20

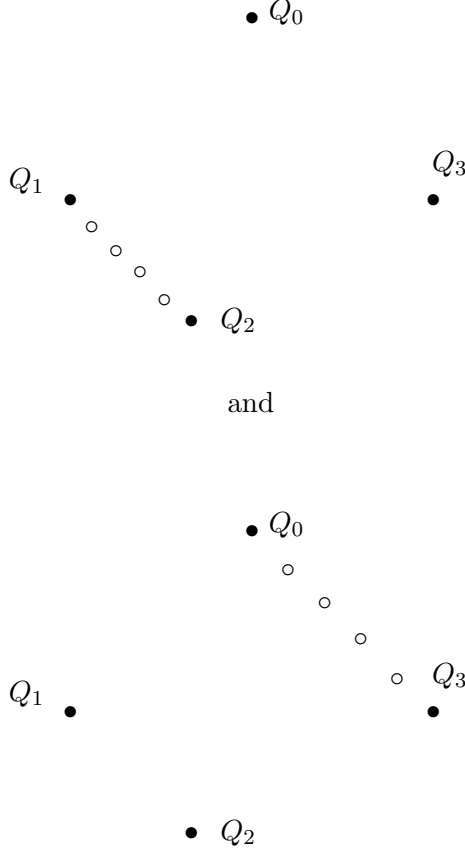
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1. INTRODUCTION

We have been studying Galois embedding of algebraic varieties [6], in particular, of elliptic curves E . In this case, by Lemma 8 in [7] we can assume the embedding is associated with the complete linear system $|nP_0|$ for some $n \geq 3$, where $P_0 \in E$. Let $f_n : E \hookrightarrow \mathbb{P}^{n-1}$ be the embedding and put $C_n = f_n(E)$. Then we consider the Galois subspaces, Galois group, the arrangement of Galois subspaces and etc. for C_n in \mathbb{P}^{n-1} . In the previous papers [1, 7] we have treated in the case where $n = 4$ and settled almost all questions. However, the arrangement of V_4 and Z_4 -lines has not been determined in sufficient detail for $j(E) = 12^3$, i.e., the curve with an automorphism of order four with a fixed point. In this article we will complete it. It needs long and tedious computations to determine the Z_4 -lines explicitly. As a byproduct we obtain the number of Galois points for an irreducible quartic curve of genus one, which is a correction of the assertion of Corollary 2 in [7].

2. STATEMENT OF RESULT

Theorem 1. *The arrangement of all the Galois lines for C_4 , where $j(C_4) = 12^3$, is illustrated by the union of the following two figures:*



In these figures, \bullet denotes the intersection of V_4 -lines and \circ denotes the intersection of a V_4 and a Z_4 -line. Four points Q_0, Q_1, Q_2 and Q_3 are not coplanar. These points form vertices of a tetrahedron. Let ℓ_{ij} be the line passing through Q_i and Q_j ($0 \leq i < j \leq 3$). Then, all the V_4 -lines are $\ell_{01}, \ell_{02}, \ell_{03}, \ell_{12}, \ell_{13}$ and ℓ_{23} . Except these lines, each line is a Z_4 -line. For each vertex there exist two Z_4 -lines passing through it. Two Z_4 -lines which do not pass through the same vertex are disjoint. A Z_4 -line meets V_4 -lines at two points as is shown above. If the one is the vertex Q_i , then we let the other be R_{ij} , where $0 \leq i \leq 3$ and $j = 1, 2$. By taking a suitable coordinates of \mathbb{P}^3 , we can give the coordinates of Q_i and R_{ij} explicitly as follows, in the following we use the notation $i = \sqrt{-1}$:

$$\begin{aligned}
 Q_0 &= (0 : 0 : 0 : 1), \quad Q_1 = (4 : -1 : 2 : 0), \quad Q_2 = (4 : -1 : -2 : 0), \\
 Q_3 &= (4 : 1 : 0 : 0), \\
 R_{01} &= (0 : 0 : 1 : 0), \quad R_{02} = (4 : -1 : 0 : 0), \quad R_{31} = (4 : -1 : 2i : 0), \\
 R_{32} &= (4 : -1 : -2i : 0), \quad R_{21} = (4 : 1 : 0 : -2\sqrt{2}i), \quad R_{22} = (4 : 1 : 0 : 2\sqrt{2}i), \\
 R_{11} &= (4 : 1 : 0 : 2\sqrt{2}), \quad R_{12} = (4 : 1 : 0 : -2\sqrt{2})
 \end{aligned}$$

In Corollary 2 in [7] we must assume $j(E) \neq 12^3$. So we correct the corollary as follows:

Corollary 2. *Let Γ be an irreducible quartic curve in \mathbb{P}^2 and E the normalization of it. Assume the genus of E is one. If $j(E) = 12^3$ (resp. $\neq 12^3$), then the number of Galois points is at most two (resp. one).*

In fact, Takahashi found the curve defined by: $s^4 + s^2u^2 + t^4 = 0$. It is easy to see that the genus of the normalization is one and $(s : t : u) = (0 : 1 : 0)$ is a Z_4 -point and $(1 : 0 : 0)$ is a V_4 -point. By using Theorem 1, we can find many such examples as follows:

Example 3. Let L_{ij} and ℓ_{pq} be the Z_4 and V_4 -lines passing through R_{ij} , where $0 \leq i \leq 3$, $j = 1, 2$ and if $i = 0$ or 3 (resp. 1 or 2), then $(p, q) = (1, 2)$ (resp. $(0, 3)$). Let $\pi_{ij} : \mathbb{P}^3 \cdots \rightarrow \mathbb{P}^2$ be the projection with the center R_{ij} . Then, $\pi_{ij}(C_4) = \Gamma_{ij}$ is an irreducible quartic curve and the points $\pi_{ij}(L_{ij})$ and $\pi_{ij}(\ell_{pq})$ are Z_4 and V_4 -points, respectively. For example, take the point $R = (0 : 0 : 1 : 0)$ as the projection center. Then, $\pi_R(X : Y : Z : W) = (X : Y : W)$. The Z_4 -line $L : X = Y = 0$ and V_4 -line $\ell : X + 4Y = W = 0$ pass through R . The defining equation of $\pi_R(C_4)$ is $W^4 = XY(X - 4Y)^2$, $\pi_R(L) = (0 : 0 : 1)$ and $\pi_R(\ell) = (-4 : 1 : 0)$. By the projective change of coordinates

$$X = X' - iY', \quad Y = -(X' + iY')/4$$

we get the example of Takahashi.

We have an interest in the group generated by the Galois groups belonging to Galois points [3, 5]. In the current case we have the following:

Let \mathcal{G}_0 (resp. \mathcal{G}) be the group generated by the Galois group belonging to V_4 -lines (V_4 or Z_4 -line) for C . Then we have the followig.

Corollary 4. (1) *In case $j \neq 12^3$, we have $\mathcal{G} = \mathcal{G}_0 = \langle \rho_0, \rho_1, \rho_2 \rangle \cong Z_2 \times Z_2 \times Z_2$. an example of the curve with this group is given in [4]*

$$(4y^4 + 5xy^2 - 1)^2 = xy^2(x + 8y^2)^2.$$

(2) *In case $j = 12^3$ we can show $\mathcal{G} = \langle \sigma_0, \sigma_2, \sigma_6 \rangle$. Putting*

$$\alpha(z) = z + \frac{1}{2}, \quad \beta(z) = z + \frac{3+i}{4},$$

we have $\langle \alpha, \beta \rangle \cong Z_2 \times Z_4$ and

$$\mathcal{G} \cong \langle \alpha, \beta \rangle \rtimes \langle \sigma_0 \rangle$$

It is easy to see that \mathcal{G}_0 is a normal subgroup of \mathcal{G} . In particular $|\mathcal{G}| = 32$ and \mathcal{G} is called an elliptic exceptional group $E(2, 2, 4)$ in [4]. Furthermore this group appears as the group by the embedding of degree 32 of the elliptic curve $j(E) = 1$.

3. PROOF

Hereafter we treat only the case $j(E) = 12^3$. We use the same notation and convention as in [7]. Let us recall briefly:

- $\pi : \mathbb{C} \rightarrow E = \mathbb{C}/\mathcal{L}$, $\mathcal{L} = \mathbb{Z} + \mathbb{Z}i$, $i = \sqrt{-1}$
- $x = \wp(z)$, $y = \wp'(z)$, \wp -functions with respect to \mathcal{L} .

- $\varphi : \mathbb{C} \longrightarrow \mathbb{C}/\mathcal{L} \xrightarrow{\sim} E : y^2 = 4x^3 - x$
- $P_\alpha := \varphi(\alpha) \in E$, ($\alpha \in \mathbb{C}$), in particular, $P_0 = \varphi(0)$
- $+$ denotes the sum of complex numbers $\alpha + \beta$ in \mathbb{C} and at the same time the sum of divisors $P_\alpha + P_\beta$ on E
- \sim : linear equivalence
- Note that $P_\alpha + P_\beta \sim P_{\alpha+\beta} + P_0$ holds true.
- V_4 : Klein's four group
- Z_n : cyclic group of order n
- $\langle \cdots \rangle$: the group generated by \cdots

Since the embedding is associated with $|4P_0|$, we can assume it is given by

$$f = f_4 : E \longrightarrow \mathbb{P}^3, \quad f(x, y) = (1 : x^2 : x : y)$$

Put $C = f(E)$. The V_4 -lines have been determined in [7]. Recall that the Galois group associated with V_4 -line is $\langle \rho_i, \rho_j \rangle$ for some i, j where $0 \leq i < j \leq 3$. Let σ be a complex representation of a generator of the group associated with Z_4 -line. As we see in the proof of Lemma 20 in [7], σ can be expressed as $\sigma(z) = iz + (m + ni)/4$, where $(m, n) = (0, 0), (2, 2), (3, 1), (1, 3), (1, 1), (3, 3), (2, 0)$ or $(0, 2)$. So we put as follows:

$$\begin{array}{ll} (0) \quad \sigma_0(z) = iz & (1) \quad \sigma_1(z) = iz + \frac{1+i}{2} \\ (2) \quad \sigma_2(z) = iz + \frac{3+i}{4} & (3) \quad \sigma_3(z) = iz + \frac{1+3i}{4} \\ (4) \quad \sigma_4(z) = iz + \frac{1+i}{4} & (5) \quad \sigma_5(z) = iz + \frac{3+3i}{4} \\ (6) \quad \sigma_6(z) = iz + \frac{1}{2} & (7) \quad \sigma_7(z) = iz + \frac{i}{2} \end{array}$$

Furthermore we put

$$\rho_0(z) = -z, \quad \rho_1(z) = -z + \frac{1}{2}, \quad \rho_2(z) = -z + \frac{i}{2}, \quad \rho_3(z) = -z + \frac{1+i}{2}.$$

Note that

$$\rho_0 \equiv \sigma_0^2 \equiv \sigma_1^2 (\text{mod } \mathcal{L}), \quad \rho_1 \equiv \sigma_2^2 \equiv \sigma_3^2 (\text{mod } \mathcal{L}),$$

$$\rho_2 \equiv \sigma_4^2 \equiv \sigma_5^2 (\text{mod } \mathcal{L}), \quad \rho_3 \equiv \sigma_6^2 \equiv \sigma_7^2 (\text{mod } \mathcal{L}).$$

Let V be the vector space spanned by $\{1, x^2, x, y\}$ over \mathbb{C} . If σ is an element of the Galois group associated with a Galois line ℓ , then it induces a linear transformation $M(\sigma)$ of V . The $M(\sigma)$ defines a projective transformation, we denote it by the same letter. It has the following properties:

- (1) Some eigenvalue belongs to at least two independent eigenvectors.
- (2) We have $M(\sigma)(\ell) = \ell$, i.e., $M(\sigma)$ induces an automorphism of $\ell \cong \mathbb{P}^1$.

There are two characterizations for the vertices, one is the following Lemma 17 in [7]:

Lemma 5. *There exist exactly four irreducible quadratic surfaces S_i ($0 \leq i \leq 3$) such that each S_i has a singular point and contains C . Let Q_i be the unique singular point of S_i . Then the four points are not coplanar.*

The other one is as follows:

Lemma 6. *The $M(\rho_i)$ ($0 \leq i \leq 3$) has two eigenvalues λ_{i1} and λ_{i2} which belong to one and three independent eigenvectors, respectively. Let Q_i be the point in \mathbb{P}^3 defined by the eigenvector having the eigenvalue λ_{i1} . Then, these points coincide with the ones in Lemma 1. The line passing through Q_i and Q_j ($0 \leq i < j \leq 3$) is a V_4 -line. Four points $\{Q_1, Q_2, Q_3, Q_4\}$ are not coplanar, so they form a vertex of a tetrahedron.*

Proof. These are checked by direct computations. To find the action of ρ_i on the vector space V , we can use the action on $x = \wp(z)$ and $y = \wp'(z)$. Making use of the addition formula on the elliptic curve, we obtain the following.

$$\begin{aligned} \rho_0(1, x^2, x, y) &= (1, x^2, x, -y) \\ \rho_1(1, x^2, x, y) &= (4x^2 - 4x + 1, x^2 + x + \frac{1}{4}, 2x^2 - \frac{1}{2}, 2y) \\ \rho_2(1, x^2, x, y) &= (4x^2 + 4x + 1, x^2 - x + \frac{1}{4}, -2x^2 + \frac{1}{2}, 2y) \\ \rho_3(1, x^2, x, y) &= (4x^2, \frac{1}{4}, -x, -y) \end{aligned}$$

We obtain the following representation matrices:

$$M(\rho_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad M(\rho_1) = \begin{pmatrix} 1 & 4 & -4 & 0 \\ 1/4 & 1 & 1 & 0 \\ -1/2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$M(\rho_2) = \begin{pmatrix} 1 & 4 & 4 & 0 \\ 1/4 & 1 & -1 & 0 \\ 1/2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad M(\rho_3) = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Therefore, the eigenvalues λ and eigenvectors (mod constant multiplications) of $M(\rho)$ can be computed as follows :

$$\begin{array}{llll} M(\rho_0) & \lambda = -1 & : & (0, 0, 0, 1) \quad \lambda = 1 \quad : \quad (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) \\ M(\rho_1) & \lambda = -2 & : & (4, -1, 2, 0) \quad \lambda = 2 \quad : \quad (1, 0, -1/4, 0), (0, 1, 1, 0), (0, 0, 0, 1) \\ M(\rho_2) & \lambda = -2 & : & (4, -1, -2, 0) \quad \lambda = 2 \quad : \quad (4, 0, 1, 0), (0, 1, -1, 0), (0, 0, 0, 1) \\ M(\rho_3) & \lambda = 4 & : & (4, 1, 0, 0) \quad \lambda = -4 \quad : \quad (4, -1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1) \end{array}$$

□

Similarly, we can find Z_4 -lines by the following results :

$$\begin{aligned}
\sigma_0(1, x^2, x, y) &= (1, x^2, -x, iy) \\
\sigma_1(1, x^2, x, y) &= (4x^2, \frac{1}{4}, x, ix) \\
\sigma_2(1, x^2, x, y) &= (-2y + \sqrt{2}(i-1)x^2 - \sqrt{2}(1+i)x - \frac{\sqrt{2}(i-1)}{4}, \\
&\quad -\frac{1}{2}y - \frac{\sqrt{2}(i-1)}{4}x^2 + \frac{\sqrt{2}(i+1)}{4}x + \frac{\sqrt{2}(i-1)}{16}, \\
&\quad \frac{\sqrt{2}(1+i)}{2}x^2 - \frac{\sqrt{2}(i-1)}{2}x - \frac{\sqrt{2}(1+i)}{8}, \\
&\quad 2x^2 + \frac{1}{2}) \\
\sigma_3(1, x^2, x, y) &= (4\sqrt{2}iy - (1+i)(4x^2 + 4ix - 1), \\
&\quad \frac{1}{4}(4\sqrt{2}iy + (1+i)(4x^2 + 4ix - 1)), \\
&\quad \frac{i-1}{2}(4x^2 - 4ix - 1), \\
&\quad -\sqrt{2}i(4x^2 + 1)) \\
\sigma_4(1, x^2, x, y) &= (-2\sqrt{2}(1+i)y - 4ix^2 - 4x + i, \\
&\quad \frac{-1-i}{\sqrt{2}}y + ix^2 + x - \frac{i}{4}, \\
&\quad 2x^2 + 2ix - \frac{1}{2}, \\
&\quad -2\sqrt{2}(1+i)x^2 - \frac{1+i}{\sqrt{2}}) \\
\sigma_5(1, x^2, x, y) &= (2\sqrt{2}(1+i)y - 4ix^2 - 4x + i, \\
&\quad \frac{1+i}{\sqrt{2}}y + ix^2 + x - \frac{i}{4}, \\
&\quad 2x^2 + 2ix - \frac{1}{2}, \\
&\quad \frac{1+i}{\sqrt{2}}(4x^2 + 1)) \\
\sigma_6(1, x^2, x, y) &= (4x^2 + 4x + 1, x^2 - x + \frac{1}{4}, 2x^2 - \frac{1}{2}, -2iy) \\
\sigma_7(1, x^2, x, y) &= (4x^2 - 4x + 1, x^2 + x + \frac{1}{4}, -2x^2 + \frac{1}{2}, -2iy)
\end{aligned}$$

$$M(\sigma_0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad M(\sigma_1) = \begin{pmatrix} 0 & 4 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

$$M(\sigma_2) = -\sqrt{2}i \begin{pmatrix} (i+1)/4 & -i-1 & -i+1 & -\sqrt{2}i \\ -(1+i)/16 & (1+i)/4 & (-1+i)/4 & -i/2\sqrt{2} \\ (1-i)/8 & (-1+i)/2 & (1+i)/2 & 0 \\ i/2\sqrt{2} & \sqrt{2}i & 0 & 0 \end{pmatrix}$$

$$M(\sigma_3) = \begin{pmatrix} 1+i & -4(1+i) & 4(1-i) & -4\sqrt{2}i \\ -(1+i)/4 & 1+i & -1+i & \sqrt{2}i \\ (1-i)/2 & -2(1-i) & 2(1+i) & 0 \\ -\sqrt{2}i & -4\sqrt{2}i & 0 & 0 \end{pmatrix}$$

$$M(\sigma_4) = \begin{pmatrix} i & -4i & -4 & -2\sqrt{2}(1+i) \\ -i/4 & i & 1 & -(1+i)\sqrt{2} \\ -1/2 & 2 & 2i & 0 \\ -(1+i)/\sqrt{2} & 2\sqrt{2}(1+i) & 0 & 0 \end{pmatrix}$$

$$M(\sigma_5) = \begin{pmatrix} i & -4i & -4 & 2\sqrt{2}(1+i) \\ -i/4 & i & 1 & (1+i)\sqrt{2} \\ -1/2 & 2 & 2i & 0 \\ (1+i)/\sqrt{2} & 2\sqrt{2}(1+i) & 0 & 0 \end{pmatrix}$$

$$M(\sigma_6) = \begin{pmatrix} 1 & 4 & 4 & 0 \\ 1/4 & 1 & -1 & 0 \\ -1/2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix} \quad M(\sigma_7) = \begin{pmatrix} 1 & 4 & -4 & 0 \\ 1/4 & 1 & 1 & 0 \\ 1/2 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2i \end{pmatrix}$$

Eigenvalues λ and eigenvectors (mod constant multiplications) of $M(\sigma)$ are as follows :

$M(\sigma_0)$	$\lambda = -1$: $(0, 0, 0, 1)$
	$\lambda = 1$: $(1, 0, 0, 0), (0, 1, 0, 0)$
	$\lambda = i$: $(0, 0, 0, 1)$
$M(\sigma_1)$	$\lambda = -1$: $(4, -1, 0, 0)$
	$\lambda = 1$: $(4, 1, 0, 0), (0, 0, 1, 0)$
	$\lambda = i$: $(0, 0, 0, 1)$
$M(\sigma_2)$	$\lambda = \sqrt{2}$: $(4, -1, -2, 0)$
	$\lambda = -\sqrt{2}i$: $(4, 0, 1, \sqrt{2}i), (0, 1, -1, \sqrt{2}i)$
	$\lambda = \sqrt{2}i$: $(4, 1, 0, -2\sqrt{2}i)$
$M(\sigma_3)$	$\lambda = 4i$: $(4, -1, -2, 0)$
	$\lambda = 4$: $(4, 0, 1, -2\sqrt{2}i), (0, 1, -1, -\sqrt{2}i)$
	$\lambda = -4$: $(4, 1, 0, 2\sqrt{2}i)$
$M(\sigma_4)$	$\lambda = -2 - 2i$: $(4, 1, 0, 2\sqrt{2})$
	$\lambda = 2 + 2i$: $(4, 0, -1, -\sqrt{2}), (0, 1, 1, \sqrt{2})$
	$\lambda = -2 + 2i$: $(4, -1, 2, 0)$
$M(\sigma_5)$	$\lambda = -2 - 2i$: $(4, 1, 0, -2\sqrt{2})$
	$\lambda = 2 + 2i$: $(4, 0, -1, \sqrt{2}), (0, 1, 1, \sqrt{2})$
	$\lambda = -2 + 2i$: $(4, -1, 2, 0)$
$M(\sigma_6)$	$\lambda = 2i$: $(4, -1, 2i, 0)$
	$\lambda = -2i$: $(4, -1, -2, 0), (0, 0, 0, 1)$
	$\lambda = 2$: $(4, 1, 0, 0)$
$M(\sigma_7)$	$\lambda = 2i$: $(4, -1, -2i, 0)$
	$\lambda = -2i$: $(4, -1, 2i, 0), (0, 0, 0, 1)$
	$\lambda = 2$: $(4, 1, 0, 0)$

The proof of Corollary 2 is the same as Corollary 2 in [7]. It is sufficient to note the intersection points of Galois lines. In the case where $j(E) = 12^3$, there exist points which are not the vertices Q_i ($0 \leq i \leq 3$) but the intersection of V_4 and Z_4 -lines. The projection from such points yield the curve with two Galois points.

4. GENERATED GALOIS GROUP

We have studied the group generated by the Galois group belonging to Galois points [3, 5]. In the case of Galois embedding of elliptic curves, we have the following.

Remark 7. For each Galois embedding let \mathcal{G} be the group generated by the Galois groups belonging to the Galois subspaces. Then \mathcal{G} can be realized as the Galois group for some Galois embedding of the elliptic curve.

Proof. We infer readily the theorem from Theorems 7.4 and 7.7 in [4]. \square

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